

Schwinger's Result On Particle Production From Complex Paths WKB Approximation

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Abstract

This paper presents the derivation of Schwinger's gauge invariant result of $Im \mathcal{L}_{eff}$ upto one loop approximation, for particle production in an uniform electric field through the method of complex trajectory WKB approximation (CWKB). The CWKB proposed by one of the author's [1] looks upon particle production as due to the motion of a particle in complex space-time plane, thereby requiring tunneling paths both in space and time. Recently [2, 3] there have been some efforts to calculate the reflection and transmission coefficients for particle production in uniform electric field that differ from our expressions for the same. In this paper we clarify the confusion in this regard and establish the correctness of CWKB.

1 Introduction

The complex trajectory WKB approximation has been an effective tool to understand particle production. It works in Minkowski and as well as in curved spacetime; both in space and time dependent gauge. Though originally proposed and developed by one of the authors [1], the method is now being pursued by some workers [2, 3, 4], technically being the same in approach with CWKB, having a slight differences with respect to interpretation and results. To show the exactness of CWKB with the standard results, we consider the particle production in constant electric field, where gauge invariant result due to Schwinger [5] is available for comparison.

Schwinger [5] demonstrated that the probability for a vacuum to remain a vacuum in a volume $L^3 = V$ over time T is determined by the quantity $\exp(-2Im \mathcal{L}_{eff})VT$, i.e., $2Im \mathcal{L}_{eff}$ is the rate of decay of the vacuum per unit 4-volume. In one loop

approximation, the Schwinger's result for a constant and uniform electrical field is

$$2Im \mathcal{L}_{eff}^{(1)} = (2s + 1) \frac{(qE_0)^2}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{(\pm)^{n+1}}{n^2} e^{\frac{-\pi n m^2}{qE_0}}. \quad (1)$$

It is a gauge invariant result. Here m is the mass of the particle produced with charge q (for fermions ($s = 1/2$) the upper sign, and for bosons ($s = 0$) the lower sign is to be taken in Eq. (1)), and E_0 is the electric field. The expression (1) is related to pair production through Nikishov's [6] representation :

$$2Im \mathcal{L}_{eff}^{(1)} L^3 T = \mp \sum_s \int \frac{d^3 k}{(2\pi)^3} L^3 \ln(1 \pm \bar{n}_p), \quad (2)$$

$$\bar{n}_p = \exp\left(-\pi \frac{m^2 + k_{\perp}^2}{qE_0}\right), \quad (3)$$

where \bar{n}_p is the average number of pairs formed by the field in the state with a given momenta and projection of spin $k = \vec{k}, s$.

The above technique of obtaining the $Im \mathcal{L}_{eff}$ can be generalized to arbitrary electromagnetic and gravitational backgrounds but is an uphill task since it is not at all convenient to regularize the effective Lagrangian in all such cases. Due to these reasons many workers in this field turn towards the Bogolubov transformation technique based on the method of normal mode analysis to understand the phenomenon of particle production. Here the problem arises with respect to interpretation when one tries to obtain the gauge invariant results, mentioned in (1) and (2). In time dependent gauge the instability of the vacuum is quite understandable. The vacuum $|0_- \rangle$ at $t \rightarrow -\infty$ is not the same as the vacuum $|0_+ \rangle$ at $t \rightarrow +\infty$. The vacuum decay $|0_- \rangle \rightarrow |0_+ \rangle$ allows one to see particle production in which the Bogolubov coefficient β_{λ} (λ is a parameter that specifies the state of produced particles) is not zero and is interpreted as

$$\langle 0_- | N(\lambda) | 0_+ \rangle = \sum_{\lambda} |\beta_{\lambda}|^2, \quad (4)$$

where $N(\lambda)$ is the number of 'in' u_i -mode particles found in the out vacuum $|0_+ \rangle$, where $u_i(\lambda)$ modes construct the in-vacuum $|0_- \rangle$ and $\bar{u}_i(\lambda)$ modes construct the out-vacuum $|0_+ \rangle$. When one considers the same problem in space dependent gauge; the vacuua at $x \rightarrow \mp\infty$ remain the same and there is no vacuum instability. Consequently one expects no particle production in a space dependent gauge. However through mode analysis one still recovers a non-zero $|\beta_{\lambda}|^2$ even in space dependent gauge, implying particle production. A tunneling interpretation is then invoked to interpret particle production in space dependent gauge because of non-zero $|\beta_{\lambda}|^2$. In this interpretation one switches to the interpretation of 'tunneling through the barrier' [4] to calculate the reflection and transmission amplitude R and T and to obtain $|\beta_{\lambda}|^2$ through a prescription, identifying $|\alpha_{\lambda}|^2 \pm |\beta_{\lambda}|^2 = 1$ as being equivalent

to $|T|^2 \mp |R|^2 = 1$ (\pm refer to spinor/scalar fields). The prescription that relates R and T with α_λ and β_λ is found to work differently in space and time dependent gauges. Recently a complex path analysis based on a technique of Landau [7] is being used [2, 3] to obtain β_λ from the expressions of R and T . In CWKB, we used the technique of ref. [8] using the idea of Cornwall and Tiktopoulos [9]. As pointed by us earlier, this has now led many to realize the usefulness of complex path analysis to understand particle production. In our view, the analytic continuation inherent in Bogolubov's technique is exploited in ref. [4] with an advantage towards the point of interpretation. However these works do not clarify the source of instability, how and where the particle production occurs. In these works [2, 3, 4] one also requires an advance knowledge of the gauge invariant result to settle the relation between the pairs (R, T) and $(\alpha_\lambda, \beta_\lambda)$. The knowledge of mode solutions for the analytic continuation $x \rightarrow -\infty$ to $x \rightarrow +\infty$ (or, $t \rightarrow -\infty$ to $t \rightarrow +\infty$) is a basic requirement in these works. In principle one should obtain $N(\lambda)$, without connecting it to β_λ , from the results of R and T themselves. There is another point of significant argument that has motivated the present work. In CWKB, the results of R and T do not coincide with ref. [2, 3, 4] and hence needs discussion. In ref. [2, 3], attempt has been made to obtain the results (1) and (2), but being partially successful. In the present paper we use CWKB to obtain the reflection amplitude R_{CWKB} and transmission amplitude T_{CWKB} and use a first principle derivation of $N(\lambda)$ without reverting to the calculation of Bogolubov co-efficient and obtain the exact results, term by term, of (1) and (2) along with the prefactor in the expression (1). In CWKB, we do not require to know the mode solutions at $t \rightarrow \pm\infty$ (or $x \rightarrow \pm\infty$). It is observed that R_{CWKB} is related to the full S -matrix elements whereas R_C , in the works of ref. [4] is related to the connected S -matrix element.

In CWKB, the particle production is considered as a 'reflection' process either in time (in time dependent gauge) or in space (in space dependent gauge). This is the well known Klein paradox like situation, found in all standard textbooks where the pair creation occurs in a barrier region due to reflection causing Zitterbewegung within wavepacket in the barrier region. Only when the potential is capable of transferring energy at least $2mc^2$ to the wave packet, i.e., the minimum energy required to conserve energy in making a pair, will then real "permanent" particles be created with the packet moving past the potential. The Zitterbewegung region is specified by two points within which the mixing of positive and negative frequency parts takes place and the interference term varies in time with a frequency at least $2mc^2$. The 'two points' are the turning points from where the particle turns back. The turning points for a given problem is fixed by the one dimensional Schroedinger-like equation either in space or in time depending on the gauge chosen.

The present work is motivated to deal with particle production through the technique of complex trajectory WKB (CWKB) approximation, proposed by us in various earlier works [10, 11, 12, 13] (mainly restricted to particle production in expanding spacetime). The organization of the paper is as follows. In section II we discuss the particle production as a process of reflection in time. It is a Klein-

Paradox-like situation not in space but in time. We choose an idealised example to elucidate our approach. In section III we show a method to calculate the number of produced particles in terms of reflection coefficient and obtain the Schwinger's result. In section IV we obtain the relation connecting the Bogolubov coefficients and the reflection coefficients. The concluding section synthesizes the results obtained in the present work and clarify some misconception on the CWKB results.

2 Basic Principles of CWKB

The method of CWKB in the context of cosmology was proposed by one of the authors in [1] and subsequently developed in [10, 11, 12, 13, 14, 15] to treat particle production in Robertson-Walker spacetime and deSitter spacetime. In Feynman spacetime diagram a pair production is looked upon as $Vac \rightarrow e^+ + e^-$, in which the vacuum supplies energy to create the pair $2mc^2$ such that the positron e^+ of energy E_{e+} and the electron of energy E_{e-} both move forward in time. Using Feynman-Stuckleberg prescription, the electron moving forward in time can be considered as positron of energy $-E_{e-}$ moving backward in time. The net result is that the pair production is looked upon as a process of reflection, not in space but in time from vacuum [9]. The change $-E_{e-}$ to $+E_{e-}$ requires that the system showing pair production through process of reflection must have turning points in time and as the system evolves from $t \rightarrow -\infty$ to $t \rightarrow +\infty$, there is rotation of currents showing antiparticle to particle rotation. Obviously the turning points are complex in time, reflecting the situation of 'over the barrier reflection'.

To get an idea of the above mechanism consider an one dimensional Schroedinger like equation in time

$$\frac{d^2\psi}{dt^2} + \omega^2(t)\psi = 0. \quad (5)$$

Eq.(5) has turning points given by

$$\omega^2(t_{1,2}) = 0 \quad (6)$$

where the turning points $t_{1,2}$ are complex. Balian and Bloch [16] formulated an approach to quantum mechanics starting from classical trajectories with complex coordinates. In the approach, within the framework of WKB approximation, wave optics is generalized to complex trajectories to build up the wave. Such a method accounts for contribution of $\exp(-\frac{\epsilon}{\hbar})$ to the usual WKB wave and is found to reproduce quantitatively all quantum mechanical effects even in cases where the scattering potential varies rapidly over a distance of wavelength or less. We applied this technique following [8], to obtain an expression for $\psi(t)$ at $t \rightarrow \infty$ as

$$\psi(t) \xrightarrow{t \rightarrow \infty} \frac{1}{(\omega(t))^{\frac{1}{2}}} \left[e^{iS(t,t_0)} - iRe^{-iS(t,t_0)} \right] \quad (7)$$

where

$$S(t, t_0) = \int_{t_0}^t \omega(t) dt \quad (8)$$

and

$$R = \frac{\exp[2iS(t_1, t_0)]}{1 + \exp[2iS(t_1, t_2)]}. \quad (9)$$

The detailed derivation of Eq. (7) will be found in [8] and also in our previous works [12, 13]. The interpretation of Eq. (7) is as follows. A wave starts at $t_0 > t$ and moving leftward arrives at t . This is represented by the first term in Eq. (7). Another wave starts from t_0 , moving leftward arrives at the turning point t_1 to get reflected to arrive back at t . This is a reflected wave contribution given by

$$\exp[+iS(t_1, t_0) - iS(t, t_1)] = \exp[2iS(t, t_0) - iS(t, t_0)]. \quad (10)$$

The contribution (10) is then multiplied by repeated reflections between the complex turning points t_1 and t_2 and its contribution is

$$\sum_{\mu=0}^{\infty} [-i \exp\{iS(t_1, t_2)\}]^{2\mu} = \frac{1}{1 + \exp[2iS(t_1, t_2)]}. \quad (11)$$

The contribution (10) times (11) give the second term in Eq. (7) as the reflected wave where R is the reflection amplitude. The essence of Eq. (7) is that there is no particle at $t \rightarrow -\infty$ i.e.,

$$\psi_{in} \xrightarrow[t \rightarrow -\infty]{} \exp[iS(t, t_0)], \quad t < 0 \quad (12)$$

but at $t \rightarrow +\infty$, Eq. (12) evolves into

$$\psi_{in} \xrightarrow[t \rightarrow +\infty]{} \frac{1}{(\omega(t))^{1/2}} [\exp[iS(t, t_0) - iR[-iS(t, t_0)]]]. \quad (13)$$

We identify R as pair production amplitude. The above method works also in space dependent gauge in which we are to replace Eq. (12) and Eq. (13) as

$$\psi_{in}(x) \xrightarrow[x \rightarrow \infty]{} e^{-iS(x, x_0)}. \quad (14)$$

$$\psi_{in}(x) \xrightarrow[x \rightarrow -\infty]{} e^{-iS(x, x_0)} - iR e^{+iS(x, x_0)}. \quad (15)$$

We applied this technique for particle production by external electromagnetic background for $A_\mu = (E_0 x, 0, 0, 0)$ as follows. The Klein-Gordon equation is

$$((\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu) + m^2)\Phi = 0. \quad (16)$$

With $M = k_x^2 + k_y^2 + m^2 = k_\perp^2 + m^2$ and

$$\Phi = e^{-i\omega t} e^{ik_x x + ik_y y + ik_z z} \phi(x), \quad (17)$$

we find from Eq. (16)

$$\frac{\partial^2 \phi}{\partial x^2} + [(\omega + qE_0)^2 - M^2] \phi = 0. \quad (18)$$

Let us substitute

$$\rho = \sqrt{qE_0}x + \frac{\omega}{q\sqrt{E_0}}, \quad (19)$$

$$\lambda = \frac{k_x^2 + k_y^2 + m^2}{qE_0}, \quad (20)$$

in Eq. (18) to get

$$\frac{\partial^2 \phi}{\partial x^2} + (\rho^2 - \lambda)\phi = 0, \quad (21)$$

so that the turning points are at $\rho = \pm\sqrt{\lambda}$. We evaluate R using Eq. (9) and Eq. (20) as

$$|R|^2 = \frac{e^{-\pi\lambda}}{(1 + e^{-\pi\lambda})^2}. \quad (22)$$

We evaluated [1] the same problem in time dependent gauge, found complex turning points in complex time plane to obtain the same expression of R . Thus the reflection amplitude R in CWKB is a gauge independent result.

Recently [2, 3] a complex path analysis similar to us [they used the technique of Landau and needs to know the mode solutions] obtained the result

$$|R_C|^2 = \frac{e^{-\pi\lambda}}{1 + e^{-\pi\lambda}}. \quad (23)$$

This result is also obtained in [4] but differs from our expression (22). Henceforth we call our reflection amplitude as R_{CWKB} to distinguish it from Eq. (23). In [4], the unitarity relation (that also follows from charge conservation)

$$|R_C|^2 + |T_C|^2 = 1, \quad (24)$$

is equated with

$$|\alpha_\lambda|^2 - |\beta_\lambda|^2 = 1 \quad (25)$$

(here α_λ and β_λ are Bogolubov coefficients) to obtain

$$|\beta_\lambda|^2 = \frac{|R_C|^2}{|T_C|^2} \quad (26)$$

$$|\alpha_\lambda|^2 = \frac{1}{|T_C|^2}. \quad (27)$$

No arguments are placed to obtain Eq. (26) and Eq. (27) though $|\beta_\lambda|^2 = \frac{|T_C|^2}{|R_C|^2}$ and $|\alpha_\lambda|^2 = \frac{1}{|R_C|^2}$ is also a possible choice. It is not at all clear what transpires to

connect $|R_C|^2$ and $|T_C|^2$ with $|\alpha_\lambda|^2$ and $|\beta_\lambda|^2$, wherein one can in principle determine the number of pairs formed ($\equiv |\beta_\lambda|^2$) simply from the results of $|R|^2$ only. The reason for such a choice may be that one knows a priori the result of Schwinger to settle Eqs. (26) and (27). In a recent work [17] we call into question the validity of Eqs. (26) and (27). In the present work we show that both $|R|$ and $|R_C|$ given in Eqs. (22) and (23) are correct, in which R (\equiv full S-matrix element) is related to disconnected propagator and R_C (\equiv connected S-matrix element) is related to connected propagator. We also derive the relations (26) and (27) from our result of R .

3 Reflection Coefficient And Average Number Of Pairs

The amplitude for observing a scalar particle at the point x_a and an antiparticle at x_b is given by [18, 19]

$$A = -A_0 \int d\vec{k}_a d\vec{k}_b d\sigma_a^\mu d\sigma_b^\nu f^*(k_a, x_a) \overleftrightarrow{\partial}_{\mu a} G(x_b, x_a) \overleftrightarrow{\partial}_{\nu b} f^*(k_b, x_b) \quad (28)$$

where $d\sigma^\mu$ is the element of a space-like hypersurface, ∂_μ is the derivative in the time like direction and A_0 is the amplitude for no particle production. Consider scalar particle in time dependent gauge. In such cases, the \vec{x} dependence of f and G are all phase factors and hence the integrations over $d^3x_{a,b}$ (required in evaluating the Green function $G(x_b, x_a)$) and $d^3k_{a,b}$ fix the above numbers of the particle and antiparticle to be $-\vec{k}_a = \vec{k}_b = \vec{k}$. If we look at $t_a = t_b$, we see a particle and an antiparticle on the same space-like surface with opposite momenta. Under these circumstances Eq. (28) can be written as

$$A = \int d^3k A(\vec{k}) \quad (29)$$

where $A(\vec{k})$ being the amplitude for creation of a pair with momenta \vec{k} and $-\vec{k}$. For an electromagnetic potential $A_\mu = (0, E_0 t, 0, 0)$, the Klein-Gordon equation is

$$m^2 f = (-\partial_t^2 + (\partial_x - qE_0 t)^2 + \partial_y^2 + \partial_z^2) f. \quad (30)$$

With

$$f(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k} \cdot \vec{x}} g(t), \quad (31)$$

we find

$$g = N D_{-\frac{1}{2} - i\frac{\lambda}{2}} (\sqrt{2iqE_0} (t - \frac{k_x}{qE_0})), \quad (32)$$

where

$$|N| = (2qE_0)^{-\frac{1}{4}} e^{-\frac{\pi\lambda}{8}}, \quad (33)$$

and

$$\lambda \equiv \frac{1}{qE_0}(m^2 + k_y^2 + k_z^2). \quad (34)$$

Evaluating the propagator it can be shown that $A(\vec{k})$ turns out to the form

$$|A(\vec{k})|^2 = |A_0|^2 \omega_k \quad (35)$$

with

$$\omega_k = \frac{1}{1 + e^{\pi\lambda}}. \quad (36)$$

Thus $|A(\vec{k})|$ is the probability for the creation of one pair with wave number \vec{k} , $|A_0|^2$ is the absolute probability of creating no particles in a given mode (i.e., in mode k) and ω_k is the relative probability of producing a pair if none were present in the initial state. This ω_k is the square of the connected S-matrix element [9] and is denoted by R_C .

It should be pointed out that in the complex path analysis, we could start from the relation (35) itself to determine $|A_0|^2$ and ω_k using the result of R_{CWKB} with $|A(\vec{k})|^2 \equiv |R_{\text{CWKB}}|^2$. Henceforth we omit the subscript ‘CWKB’ i.e., write $R_{\text{CWKB}} \equiv R$. It may be pointed out that R is the full S -matrix elements whereas the ω_k by definition is related only to the connected S -matrix elements so that $\omega_k = |R_C(\vec{k})|^2$ i.e., R_C is related to the connected propagator S_{AC} whereas R is related to the disconnected propagator S_A . These propagators are related by

$$S_A = e^{iW} S_{AC} \quad (37)$$

where $W = \text{Tr} \ln S_{AC}$ is the sum of connected vacuum graphs [see [9]].

The probability of producing n pairs of particles with wave number \vec{k} is then

$$P_n(\vec{k}) = |A_0|^2 \omega_k^n.$$

Hence the total probability of creating $n = 0, 1, 2, \dots$ pair will sum to unity (for Bose statistics) i.e.,

$$\begin{aligned} |A_0|^2(1 + \omega_k + \omega_k^2 + \dots) &= 1 \\ \text{or, } |A_0|^2 &= 1 - \omega_k. \end{aligned} \quad (38)$$

Hence the absolute probability for 1-pair production is

$$|A(\vec{k})|^2 \equiv P_1(\vec{k}) = (1 - \omega_k) \omega_k. \quad (39)$$

By definition the reflection coefficients $|R_{\text{CWKB}}|^2 \equiv |R|^2$ equals Eq. (39) so that

$$|R|^2 = \omega_k(1 - \omega_k). \quad (40)$$

The average number of pairs with wavenumber \vec{k} is

$$N(\vec{k}) = \sum_{n=0}^{\infty} n P_n(\vec{k}) = \frac{\omega_{\vec{k}}}{1 - \omega_{\vec{k}}}. \quad (41)$$

Let us now relate Eqs. (38), (40) and (41) to the vacuum persistence probability. By definition $|A_0|^2$ is the probability of no pair production for wave number \vec{k} . Hence the total probability of no pair production for all wave number \vec{k} is

$$\begin{aligned} \prod_{\vec{k}} |A_0(\vec{k})|^2 &= \prod_{\vec{k}} (1 - \omega_{\vec{k}}) = \prod_{\vec{k}} (1 + N(\vec{k}))^{-1} \\ &\equiv e^{-\sum_{\vec{k}} \ln(1 + N(\vec{k}))}. \end{aligned} \quad (42)$$

Eq. (42) gives the probability that no particles are created in the vacuum in any given mode i.e., the probability that the vacuum remains the vacuum is given by

$$\begin{aligned} | \langle out | in \rangle |^2 &= \prod_{\vec{k}} |A_0(\vec{k})|^2 \\ or, e^{-2Im \mathcal{L}_{eff} V_4} &= e^{-\sum_{\vec{k}} \ln(1 + N(\vec{k}))} \\ or, 2Im \mathcal{L}_{eff} V_4 &= \sum_{\vec{k}} \ln(1 + N(\vec{k})), \end{aligned} \quad (43)$$

where $V_4 = L^3 T$. In CWKB

$$|R|^2 = \frac{e^{-\pi\lambda}}{(1 + e^{-\pi\lambda})^2}. \quad (44)$$

Using Eq. (40), we find two roots

$$\omega_{\vec{k}} = \frac{1}{1 + e^{-\pi\lambda}}, \quad \frac{e^{-\pi\lambda}}{1 + e^{-\pi\lambda}}. \quad (45)$$

We take the second root so that

$$\omega_{\vec{k}} = \frac{e^{-\pi\lambda}}{1 + e^{-\pi\lambda}}. \quad (46)$$

$$1 - \omega_{\vec{k}} = \frac{1}{1 + e^{-\pi\lambda}}. \quad (47)$$

The reasons for such a choice will shortly be clear. Using the values of Eqs. (46) and (47) in Eq. (41), we get

$$N(\vec{k}) = e^{-\pi\lambda} \quad (48)$$

so that

$$2Im \mathcal{L}_{eff} V_4 = \sum_{\vec{k}} \ln(1 + e^{-\pi\lambda}). \quad (49)$$

If the electric field $E_0 \rightarrow 0$, $|out\rangle \rightarrow |in\rangle$ and there is no instability so that $\langle out|in\rangle = 1$. This implies that $Im\mathcal{L}_{eff} \rightarrow 0$. From (34), we see $\lambda = \frac{(m^2+k_\perp^2)}{qE_0} \rightarrow \infty$ when $E_0 \rightarrow 0$. Hence from Eq. (40) we have $Im\mathcal{L}_{eff} = 0$ implying $\langle out|in\rangle = 1$. Had we chosen the other root in Eq. (45), we would get $N(\vec{k}) = \exp(\pi\lambda)$ and it goes to infinity as $E_0 \rightarrow 0$, giving $\langle out|in\rangle \rightarrow infinite$. This is unphysical. Thus the correct choice is given by Eq. (46).

Expanding the log term in Eq. (43) we get

$$\begin{aligned} V_4 \cdot 2Im\mathcal{L}_{eff} &= \sum_{\vec{k},n} (-1)^{n+1} \frac{1}{n} e^{-\pi n \lambda} \\ &= \sum_{\vec{k},n} (-1)^{n+1} \frac{1}{n} e^{-\frac{\pi n (k_\perp^2 + m^2)}{qE_0}}. \end{aligned}$$

We convert the sum by integral as follows

$$\begin{aligned} 2Im\mathcal{L}_{eff} V_4 &= \sum_{n=1}^{\infty} \int dk_y \frac{L}{2\pi} \int dk_z \frac{L}{2\pi} \int_0^{qE_0 L} dk_0 \frac{T}{2\pi} e^{-\frac{\pi n (k_\perp^2 + m^2)}{qE_0}} (-1)^{n+1} \frac{1}{n} \\ &= \frac{(qE_0)^2 L^3 T}{(2\pi)^3} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} e^{-\frac{n\pi m^2}{qE_0}}. \end{aligned} \quad (50)$$

Since $V_4 = L^3 T$, we get

$$2Im\mathcal{L}_{eff} = \frac{(qE_0)^2}{(2\pi)^3} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} e^{-\frac{n\pi m^2}{qE_0}}. \quad (51)$$

This is exactly the Schwinger result for $Im\mathcal{L}_{eff}$ in one loop approximation. Eq. (43) is the Nikishov's representation (Eq. (2)). Our derivation nowhere requires to evaluate the Bogolubov co-efficient or to know the mode solutions. This amply reflects the gauge invariant content of CWKB.

4 Reflection Coefficient And Bogolubov Coefficients

Let the positive and negative frequency solutions be denoted by $\phi_{k\pm}$ (in the limit $t \rightarrow -\infty$) and

$$\Phi = \sum_{\vec{k}} (a_{\vec{k}}^{in} \phi_{\vec{k}(+)} + b_{\vec{k}}^{in} \phi_{\vec{k}(-)}), \quad (52)$$

$$a_{\vec{k}}^{in} |Vac\rangle \equiv a_{\vec{k}}^{in} |in\rangle = 0. \quad (53)$$

Eq. (53) specifies the in vacuum. For out vacuum let us define

$$\Phi = \sum_{\vec{k}} (a_{\vec{k}}^{out} \phi_{\vec{k}}^{(+)} + b_{\vec{k}}^{out} \phi_{\vec{k}}^{(-)}), \quad (54)$$

$$a_{\vec{k}}^{out}|out\rangle = 0 \quad (55)$$

where $\phi_{\vec{k}}^{(\pm)}$ are the positive and negative frequency solutions at $t \rightarrow +\infty$. The Bogolubov coefficients $\alpha_{\vec{k}} \beta_{\vec{k}}$ are now given by

$$\Phi_{\vec{k}(+)} = \alpha_{\vec{k}}\phi_{\vec{k}}^{(+)} + \beta_{\vec{k}}\phi_{\vec{k}}^{(-)}, \quad (56)$$

$$\Phi_{\vec{k}(-)} = \beta_{\vec{k}}^*\phi_{\vec{k}}^{(+)} + \alpha_{\vec{k}}^*\phi_{\vec{k}}^{(-)}, \quad (57)$$

with

$$|\alpha_{\vec{k}}|^2 - |\beta_{\vec{k}}|^2 = 1,$$

$$\phi_{\vec{k}}^{(+)} = [\phi_{\vec{k}}^{(-)}]^*, \quad \phi_{\vec{k}(+)} = [\phi_{\vec{k}(-)}]^*.$$

The operators $a_{\vec{k}}^{in}$ and $b_{\vec{k}}^{in}$ are connected to the out operators as (we now omit the vector sign over k)

$$a_k^{in} = \alpha_k^* a_k^{out} - \beta_k^* b_k^{out}, \quad (58)$$

$$b_k^{+in} = -\beta_k a_k^{out} + \alpha_k b_k^{+out}. \quad (59)$$

The pair formation amplitude is

$$a_{pair} = \langle out | a_{k' out} b_{\vec{k} out} | in \rangle. \quad (60)$$

Using (59), we write it as

$$\begin{aligned} a_{pair} &= \langle out | a_{k' out} (\alpha_k^{*-1} b_{\vec{k} in} + \frac{\beta_k^*}{\alpha_k^*} a_{k out}^+) | in \rangle \\ &= \frac{\beta_k^*}{\alpha_k^*} \langle out | a_{k' out} a_{k out}^+ | in \rangle \\ &= \frac{\beta_k^*}{\alpha_k^*} \langle out | \delta_{kk'} + a_{k out}^+ a_{k' out} | in \rangle = \frac{\beta_k^*}{\alpha_k^*} \langle out | in \rangle, \end{aligned} \quad (61)$$

since $[a_{k out}, a_{k' out}^+] = \delta_{kk'}$ and $\langle out | a_k^+ = 0$. The relative amplitude of producing a pair if none were present in the initial state is

$$\frac{\langle out | a_{k' out} b_{k out} | in \rangle}{\langle out | in \rangle}. \quad (62)$$

The square of this amplitude gives the probability of creating a pair in the given mode. It is ω_k as denoted earlier [see Eq. (36) and the discussion thereafter]. Hence, using (61) and (62)

$$\omega_k = \left| \frac{\beta_k}{\alpha_k} \right|^2 = |R_C|^2. \quad (63)$$

We thus get

$$\frac{\omega_k}{1 - \omega_k} = \frac{|\beta_k|^2}{|\alpha_k|^2 - |\beta_k|^2} = |\beta_k|^2. \quad (64)$$

Hence

$$|R_C|^2 = \frac{e^{-\pi\lambda}}{1 + e^{-\pi\lambda}}. \quad (65)$$

This result of R_C was derived in [2, 3] using the technique of Landau [7] and also in [4] by saddle point method. Let us try to understand the relation between R and R_C .

$$\begin{aligned} |R|^2 &= \frac{1}{1 + e^{-\pi\lambda}} |R_C|^2 \\ &= e^{-\ln(1+N(k))} |R_C|^2 \\ &= e^{-2Im \mathcal{L}_{eff}(k) \cdot V_4} |R_C|^2. \end{aligned} \quad (66)$$

This connection between R and R_C was also cited in ref. [9] where

$$\begin{aligned} 2Im \mathcal{L}_{eff} V_4 &= 2 \sum_k Im \mathcal{L}_{eff}(k) \cdot V_4 \\ &= \sum_k \ln(1 + N(k)). \end{aligned} \quad (67)$$

From (63), we find

$$|R_C|^2 = \frac{|\beta_k|^2}{|\alpha_k|^2}, \quad (68)$$

so that

$$|T_C|^2 = 1 - |R_C|^2 = \frac{|\alpha_k|^2 - |\beta_k|^2}{|\alpha_k|^2} = \frac{1}{|\alpha_k|^2}. \quad (69)$$

From the unitarity relation $|R_C|^2 + |T_C|^2 = 1$, we get

$$|T_C|^2 = \frac{1}{|\alpha_k|^2}. \quad (70)$$

Hence

$$|\beta_k|^2 = \frac{|R_C|^2}{|T_C|^2}, \quad |\alpha_k|^2 = \frac{1}{|T_C|^2} \quad (71)$$

The results (70) and (71) coincide with (26) and (27) as were taken in [4]. The CWKB result of R thus correctly reproduces the result of R_C obtained by other methods.

5 Conclusion

Let us try to synopsise the results. In the complex path analysis [7], that uses Landau's technique or in the saddle point method [4], one obtains R_C and T_C . These R_C and T_C are related by charge conservation as follow

$$|R_C|^2 + |T_C|^2 = 1, \quad (\text{scalar case}) \quad (72)$$

$$1 + |R_C|^2 = |T_C|^2, \quad (\textit{spinor case}). \quad (73)$$

Using the unitarity constraint of Bogolubov coefficients, one determines $|\beta_k|^2$ with the identification $N(k) = |\beta_k|^2$. Using Nikishov representation one relates $N(k)$ with $Im \mathcal{L}_{eff}$ to obtain the Schwinger result. Here one needs to know the mode solutions as well as, the continuation from $t \rightarrow -\infty (x \rightarrow -\infty)$ to $t \rightarrow +\infty (x \rightarrow +\infty)$. Moreover in [4, 7], there is an ambiguity in fixing β_k and α_k from Eqs. (72) and (73). In CWKB, we calculate R by simple integrals and particle and antiparticle states are fixed by WKB definition (Parker's definition). This fixed the vacuum. Nikishov's and Schwinger's results follow through our derivation. The instability of the vacuum that causes particle production occurs due to motion in complex-coordinate plane. There is a rotation of currents as the system evolves from $t \rightarrow -\infty (x \rightarrow -\infty)$ to $t \rightarrow +\infty (x \rightarrow +\infty)$. The mixing of positive and negative frequency states occurs between the turning points. The value of the wavefunction at real t is also contributed, not only by real trajectories but also by complex trajectories, in the sense of WKB approximation. Our result shows that CWKB justifiably takes at least one loop quantum correction within the semiclassical approximation. We have shown elsewhere in detail that the particle production in CWKB is equivalent to tunneling from left Rindler wedge to right Rindler wedge via real x and imaginary time plane. Not realizing the difference between R and R_C and the beauty of the complex path analysis, some circles argue the expression of R as wrong comparing it with R_C . We clarify our view in this respect and establish the correctness of CWKB.

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